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NATURAL MODES OF VIBRATION OF LINEAR VISCOELASTIC CIRCULAR PLATES WITH FREE EDGES

PAUL B. BAILEY and PETER J. CHEN
Sandia National Laboratories, Albuquerque, NM 87185, U.S.A.

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Abstract—In this paper we consider an extension of the classical bending theory of thin plates including the effects of mechanical dissipation. The resulting differential equation is solved for the case of circular plates with free edges. We obtain results giving the attenuation coefficients and frequencies of the natural modes of vibration of the plates. The results indicate, in particular, that when the instantaneous and equilibrium Poisson's ratios are different the values of the attenuation coefficients depend on the mode shapes, which we believe are consistent with physical notions. Otherwise, the values of the attenuation coefficients are the same for all mode shapes for a given set of material parameters.

1. INTRODUCTION

The occurrence of a mechanical resonance of a vibrating structure at a particular frequency is usually recognized experimentally by the divergence of the mechanical displacement at that frequency. For a structure with mechanical dissipation the divergence of the displacement is also accompanied by a phase shift of the displacement with respect to the driving stimulus.† The solution of the second-order differential equation describing the responses of the forced vibration of a mass–spring–damper system exhibits characteristics consistent with experimental observations. The classical bending theory of thin elastic plates due to Lagrange seems to predict fairly satisfactorily the flexural resonant frequencies and the nodal systems of such plates, but we are not aware of an extension of the theory which includes the influence of mechanical dissipation giving the attenuation coefficients and frequencies of the natural modes. There are, of course, numerous papers in the literature dealing with the subject of mechanical dissipation. For instance, the paper by Ross *et al.* [3] gives the damping effects of viscoelastic layers on plate structures. Many of the papers cited in Ref. [3] deal with similar subjects.

In this paper we consider an extension of the classical bending theory of thin plates including the influence of mechanical dissipation. We also include the effects of rotary inertia but not those of transverse shear.‡ In particular, we consider standing-wave solutions of the governing differential equation giving the attenuation coefficients and the frequencies associated with various natural modes of circular plates with free edges. As we shall see, the solution procedure requires the evaluation of Bessel functions with complex arguments. The code for this purpose has been developed quite recently at Sandia National Laboratories and is not generally available.

2. CONSTITUTIVE RELATIONS AND THE GOVERNING DIFFERENTIAL EQUATION

To begin with, let the x–y plane denote the middle surface of the plate which has thickness h. Let w denote the displacement component in the normal direction z to the

† Numerous experimental results are cited in the comprehensive report by Leissa[1]. Chen[2] gives quantitative results based on *simultaneous* measurements of the displacements of opposite surface points of electrically forced flexural mechanical resonant mode shapes of a circular disc of PLZT7/65/35, a ferroelectric ceramic. The natural modes are exhibited by the quadrature components of the displacements.

‡ Transverse shear involves the introduction of an added parameter κ, Mindlin[4], whose value is assignable based on comparison with other results. Its inclusion in the current consideration would add complications in the numerical procedure depending on the value of κ. As it turns out, the effects of rotary inertia are negligible with regard to the results given in this paper. We suspect that this will also be the case for transverse shear.

middle surface. The stress components σ_x, σ_y and σ_{xy} in terms of the strain components ε_x, ε_y and ε_{xy} are given by

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) + \sigma_x^{\mathrm{T}}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) + \sigma_y^{\mathrm{T}}$$

$$\sigma_{xy} = \frac{1-\nu}{2}\cdot\frac{E}{1-\nu^2}\varepsilon_{xy} + \sigma_{xy}^{\mathrm{T}} \tag{1}$$

where E and ν are the instantaneous Young's modulus and Poisson's ratio, respectively. σ_x^{T}, σ_y^{T} and σ_{xy}^{T} obey the rate laws†

$$\dot{\sigma}_x^{\mathrm{T}} - \alpha\sigma_x^{\mathrm{T}} = \frac{\bar{E}}{1-\bar{\nu}^2}(\varepsilon_x + \bar{\nu}\varepsilon_y)$$

$$\dot{\sigma}_y^{\mathrm{T}} - \alpha\sigma_y^{\mathrm{T}} = \frac{\bar{E}}{1-\bar{\nu}^2}(\varepsilon_y + \bar{\nu}\varepsilon_x)$$

$$\dot{\sigma}_{xy}^{\mathrm{T}} - \alpha\sigma_{xy}^{\mathrm{T}} = \frac{1-\bar{\nu}}{2}\cdot\frac{\bar{E}}{1-\bar{\nu}^2}\varepsilon_{xy} \tag{2}$$

where α is required to be negative, and $\bar{E}$ and $\bar{\nu}$ are constants. Let E_e and ν_e denote the equilibrium Young's modulus and Poisson's ratio. It follows from eqns $(1)_1$ and $(2)_1$ that at equilibrium we must have

$$\frac{E_e}{1-\nu_e^2} = \frac{E}{1-\nu^2} - \frac{\bar{E}}{\alpha(1-\bar{\nu}^2)}$$

$$\frac{E_e\nu_e}{1-\nu_e^2} = \frac{E\nu}{1-\nu^2} - \frac{\bar{E}\bar{\nu}}{\alpha(1-\bar{\nu}^2)} \tag{3}$$

which in turn implies that

$$\frac{1-\nu_e}{2}\frac{E_e}{1-\nu_e^2} = \frac{1-\nu}{2}\frac{E}{1-\nu^2} - \frac{1-\bar{\nu}}{2}\frac{\bar{E}}{\alpha(1-\bar{\nu}^2)}. \tag{4}$$

The system of eqns (1) and (2) may be regarded as the constitutive relations of a thin viscoelastic plate with

$$\tau = -\frac{1}{\alpha} \tag{5}$$

being the relaxation time.

Given the preceding constitutive relations it is possible to show that the classical

† Equations (1) and (2) are entirely equivalent to the usual integral forms of linear viscoelasticity. In general, eqns (1) and (2) are much easier to solve numerically.

notion of bending theory† and the equations of balance of linear momentum imply that, in the absence of surface traction and body force, the displacement component w which is a function of (x, y, t) obeys the differential equation

$$D\nabla^4 w + \bar{D}\int_{-\infty}^{t} \nabla^4 w(\tau)\, e^{\alpha(t-\tau)}\, d\tau - \frac{\rho h^3}{12}\nabla^2 \ddot{w} + \rho h \ddot{w} = 0 \tag{6}$$

where ∇^4 is the biharmonic operator in (x, y), ρ is the mass density of the plate material, and

$$D = \frac{Eh^3}{12(1-\nu^2)}, \qquad \bar{D} = \frac{\bar{E}h^3}{12(1-\bar{\nu}^2)}. \tag{7}$$

The governing differential equation, eqn (6), readily reduces to the classical equation of thin elastic plates, i.e. by simply deleting the term containing the integral.

In addition to the differential equation, eqn (6), we need to specify the appropriate boundary conditions. For a circular plate with a free edge and in terms of polar coordinates (r, θ, z), we have

$$\begin{aligned} &M_r = 0 \text{ at } r = a \\ &Q_r + \frac{1}{r}\frac{\partial}{\partial \theta} M_{r\theta} = 0 \text{ at } r = a \end{aligned} \tag{8}$$

where a is the radius of the plate, and

$$\begin{aligned} M_r &= \int_{-h/2}^{h/2} \sigma_r z\, dz \\ &= -D\left\{\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right\} \\ &\quad - \bar{D}\int_{-\infty}^{t}\left\{\frac{\partial^2 w(\tau)}{\partial r^2} + \bar{\nu}\left(\frac{1}{r}\frac{\partial w(\tau)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(\tau)}{\partial \theta^2}\right)\right\} e^{\alpha(t-\tau)}\, d\tau \end{aligned}$$

$$\begin{aligned} Q_r &= \int_{-h/2}^{h/2} \sigma_{rz}\, dz \\ &= -D\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + \frac{\rho h^3}{12}\frac{\partial \ddot{w}}{\partial r} \\ &\quad - \bar{D}\int_{-\infty}^{t}\left\{\frac{\partial}{\partial r}\left(\frac{\partial^2 w(\tau)}{\partial r^2} + \frac{1}{r}\frac{\partial w(\tau)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(\tau)}{\partial \theta^2}\right) e^{\alpha(t-\tau)}\, d\tau\right. \end{aligned}$$

$$\begin{aligned} M_{r\theta} &= \int_{-h/2}^{h/2} \sigma_{r\theta} z\, dz \\ &= -D(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) - \bar{D}(1-\bar{\nu})\int_{-\infty}^{t}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w(\tau)}{\partial \theta}\right) e^{\alpha(t-\tau)}\, d\tau. \end{aligned}$$

†See, e.g. Reismann and Pawlik[5], Chap. 6.

3. STANDING-WAVE SOLUTIONS

Our aim is to find a solution of the axial component w of the displacement as a function of (r, θ, t). Let

$$w = \hat{w}g \tag{9}$$

with

$$\hat{w} = \hat{w}(r, \theta), \qquad g = g(t) \tag{10}$$

and define the function G via the relation

$$G(t) = \int_{-\infty}^{t} e^{\alpha(t-\tau)} g(\tau)\,d\tau \tag{11}$$

so that

$$\dot{G} - \alpha G = g. \tag{12}$$

Substitution of the preceding into the governing differential equation yields

$$\nabla^4 \hat{w} \Big/ \left(\frac{\rho h^3}{12} \nabla^2 \hat{w} - \rho h \hat{w}\right) = \ddot{g}/(gD + G\bar{D}) = \text{constant}. \tag{13}$$

We may take this constant to be of the form

$$-\frac{k^4}{\rho h} \tag{14}$$

where k is, in general, a complex constant which is to be determined.

By eqns (13) and (14), we have, in particular

$$\nabla^4 \hat{w} + \frac{k^4 h^2}{12} \nabla^2 \hat{w} - k^4 \hat{w} = 0 \tag{15}$$

which may be rewritten in the form

$$(\nabla^2 + k_+^2)(\nabla^2 - k_-^2)\hat{w} = 0$$

with

$$2k_+^2 = \frac{k^2 h^2}{12} + k^2\left(\frac{k^4 h^4}{144} + 4\right)^{1/2}$$

$$2k_-^2 = -\frac{k^4 h^2}{12} + k^2\left(\frac{k^4 h^4}{144} + 4\right)^{1/2}. \tag{16}$$

Solutions $\hat{w}(r, \theta)$ of the differential equation, eqn (15), which are finite as $r = 0$ include

$$\hat{w} = \{AJ_n(k_+r) + BI_n(k_-r)\}\cos n\theta \tag{17}$$

where J_n and I_n are Bessel functions, A and B are, in general, complex constants, and n is, of course, an integer.

Given eqns (9), (11), (12) and (17), boundary conditions (8) become

$$\frac{E}{1-\nu^2}(Ap_n + Bq_n)g + \frac{\bar{E}}{1-\bar{\nu}^2}(A\bar{p}_n + B\bar{q}_n)G = 0 \tag{18}$$

$$\frac{E}{1-\nu^2}(Af_n + Bh_n)g + \frac{\bar{E}}{1-\bar{\nu}^2}(A\bar{f}_n + B\bar{h}_n)G + \rho(Ak_+J_n'(k_+a) + Bk_-I_n'(k_-a))\ddot{g} = 0 \tag{19}$$

where

$$\begin{aligned}
a^2p_n &= (\nu - 1)(k_+aJ_n'(k_+a) - n^2J_n(k_+a)) - k_+^2a^2J_n(k_+a)\\
a^2q_n &= (\nu - 1)(k_-aI_n'(k_-a) - n^2I_n(k_-a)) + k_-^2a^2I_n(k_-a)\\
a^3f_n &= n^2(\nu - 1)(k_+aJ_n'(k_+a) - J_n(k_+a)) - k_+^3a^3J_n'(k_+a)\\
a^3h_n &= n^2(\nu - 1)(k_-aI_n'(k_-a) - I_n(k_-a)) + k_-^3a^3I_n'(k_-a)
\end{aligned}$$

and

$$\begin{aligned}
a^2\bar{p}_n &= (\nu - 1)(k_+aJ_n'(k_+a) - n^2J_n(k_+a)) - k_+^2a^2J_n(k_+a)\\
a^2\bar{q}_n &= (\nu - 1)(k_-aI_n'(k_-a) - n^2I_n(k_-a)) + k_-^2a^2I_n(k_-a)\\
a^3\bar{f}_n &= n^2(\nu - 1)(k_+aJ_n'(k_+a) - J_n(k_+a)) - k_+^3a^3J_n'(k_+a)\\
a^3\bar{h}_n &= n^2(\nu - 1)(k_-aI_n'(k_-a) - I_n(k_-a)) + k_-^3a^3I_n'(k_-a).
\end{aligned}$$

Equation (18) implies that we have the requirement

$$\frac{g}{G} = C \tag{20}$$

where C is, in general, a complex constant. Now let

$$g = e^{\varepsilon t} \tag{21}$$

where ε is a complex constant which is to be determined. It follows from eqn (11) that

$$G = \frac{1}{\varepsilon - \alpha}e^{\varepsilon t} \tag{22}$$

provided that

$$\mathrm{Re}(\varepsilon - \alpha) > 0. \tag{23}$$

Equation (22) satisfies requirement (20). By eqns (13) and (14), we have, in particular

$$\ddot{g} + \frac{k^4}{\rho h}(gD + G\bar{D}) = 0 \tag{24}$$

which together with eqns (21) and (22) yields

$$\varepsilon^2 + \frac{k^4}{\rho h}\left(D + \frac{\bar{D}}{\varepsilon - \alpha}\right) = 0. \tag{25}$$

That is, ε must be a root of this cubic equation. Further, boundary conditions (18) and (19) reduce to

$$\frac{E}{1 - \nu^2}(Ap_n + Bq_n)(\varepsilon - \alpha) + \frac{\bar{E}}{1 - \bar{\nu}^2}(A\bar{p}_n + B\bar{q}_n) = 0 \tag{26}$$

$$\frac{E}{1 - \nu^2}(Af_n + Bh_n)(\varepsilon - \alpha) + \frac{\bar{E}}{1 - \bar{\nu}^2}(A\bar{f}_n + B\bar{h}_n) + \rho\varepsilon^2(Ak_+J_n'(k_+a) + Bk_-I_n'(k_-a))(\varepsilon - \alpha) = 0. \tag{27}$$

For the sake of brevity, eqns (26) and (27) may be rewritten in the forms

$$\begin{aligned} AC_1 + BC_2 &= 0 \\ AD_1 + BD_2 &= 0 \end{aligned} \tag{28}$$

where C_1, C_2, D_1 and D_2 are appropriately defined quantities. It follows that we must have

$$C_1D_2 - C_2D_1 = 0 \tag{29}$$

in order to satisfy the boundary conditions.

A standing-wave solution of the differential equation (6) and boundary conditions (8) is, therefore, of the form

$$w = \{AJ_n(k_+r) + BI_n(k_-r)\}\cos n\theta\, e^{\varepsilon t}. \tag{30}$$

It entails the determination of the values of ε and k such that eqns (25) and (29) are satisfied. The complex constants A and B are then determined via eqns (28). Since only the ratio of these constants is unique, it is not possible to determine the absolute magnitude of w. The nodal system of a natural mode of vibration is given by the loci of the zeros of w. For a particular mode, the real part of ε gives the attenuation coefficient, and its imaginary part is the angular frequency. It should be noted that the solution is valid only if inequality (23) is satisfied.

4. SOME REMARKS ON THE NUMERICAL TECHNIQUE

The code required to solve the problem is fairly complex. Its details are beyond the scope of this paper. We shall, however, outline the essential steps of its algorithm.

For a given nodal system, we pick a complex number for the value of k and determine the three roots ε_i, $i = 1, 2, 3$, of the cubic eqn (25). For a given pair (k, ε_i) of numbers we evaluate the appropriately defined quantities C_1, C_2, D_1 and D_2. This process is repeated until boundary conditions (26) and (27) are satisfied, or equivalently eqn (29) is satisfied. If eqn (29) is satisfied for the given pair, then we examine whether each of the other two pairs of (k, ε_i) for the same k also satisfy eqn (29) or not. If a given pair (k, ε_i) does not satisfy eqn (29), then we repeat the process for a different pair until we find the pair which does.

Given the complexities of the problem and the inherent inaccuracies of numerical computations, it would be difficult to satisfy eqn (29) *identically*. Therefore, we compute the values of the real and imaginary parts of the products C_1D_2 and C_2D_1. Equation (29) is said to be satisfied if *both* the differences

$$\mathrm{Re}(C_1D_2) - \mathrm{Re}(C_2D_1)$$

and

$$\mathrm{Im}(C_1D_2) - \mathrm{Im}(C_2D_1)$$

are minimized in magnitude. On many occasions we encounter situations when either one or the other condition is satisfied but not both. Such situations are discarded as possible solutions.

The evaluation of the quantities C_1, C_2, D_1 and D_2 requires the evaluation of Bessel functions with complex arguments. The algorithm and the code necessary for this purpose have only been recently developed by our colleague D. E. Amos here at Sandia National Laboratories. The associated paper of this development will be published in the near future.

5. NUMERICAL RESULTS

We now give the results of our computations. For our present purposes, we consider a circular plate with dimensions $a = 1.09$ and $h = 0.02$. The material properties of the plate are completely specified by the mass density ρ, the instantaneous Young's modulus E and Poisson's ratio ν, and the equilibrium Young's modulus E_e and Poisson's ratio ν_e. The quantity α is regarded as an assignable constant. Based on physical grounds, we require that

$$\frac{E_e}{1-\nu_e^2} < \frac{E}{1-\nu^2}$$

$$\frac{E_e\nu_e}{1-\nu_e^2} < \frac{E\nu}{1-\nu^2}$$

$$\frac{1-\nu_e}{2}\frac{E_e}{1-\nu_e^2} < \frac{1-\nu}{2}\frac{E}{1-\nu^2}.$$

Note that if

$$E_e < E \quad \text{and} \quad \nu_e = \nu \tag{31}$$

then the above conditions are always satisfied. On the other hand, if

$$E_e < E \quad \text{and} \quad \nu_e \neq \nu \tag{32}$$

then the above conditions place certain restrictions on the relative magnitudes of the material properties. Based on considerations in the three-dimensional context we must require that E_e is always less than E.

As it turns out, there are two classes of solutions depending entirely on whether condition (31) is satisfied or condition (32) is satisfied. The two classes of solutions are quite different. Therefore, we shall present the results of each class separately.

When condition (31) is satisfied, the value of k turns out to be real. The solution of the cubic equation (25) yields a real root and two complex roots. The real parts of the three roots are negative, and the two complex roots are complex conjugates. Let ε_0 denote the real root, and let ε_1 and $\bar{\varepsilon}_1$ denote the complex roots. All three pairs, i.e. (k, ε_0), (k, ε_1) and $(k, \bar{\varepsilon}_1)$, satisfy the boundary conditions in the sense of eqn (29). This means that for a given nodal system there are two possible solutions, namely the solution corresponding to

Table 1

s,n	k	FREQUENCY	s,n	k	FREQUENCY
1,0	2.7624	7.8891×10^{3}	0,3	3.2098	1.0651×10^{4}
2,0	5.6907	3.3479×10^{4}	1,3	6.6705	4.6001×10^{4}
3,0	8.5901	7.6285×10^{4}	2,3	9.6952	9.7176×10^{4}
4,0	11.475	1.3614×10^{5}	3,3	12.648	1.6538×10^{5}
5,0	14.351	2.1291×10^{5}	4,3	15.566	2.5048×10^{5}
1,1	4.1542	1.7841×10^{4}	5,3	18.461	3.5233×10^{5}
2,1	7.0939	5.2025×10^{4}	0,4	4.2557	1.8723×10^{4}
3,1	9.9977	1.0333×10^{5}	1,4	7.8534	6.3762×10^{4}
4,1	12.885	1.7164×10^{5}	2,4	10.931	1.2354×10^{5}
5,1	15.761	2.5682×10^{5}	3,4	13.916	2.0020×10^{5}
0,2	2.1043	4.5779×10^{3}	4,4	16.854	2.9368×10^{5}
1,2	5.4443	3.0643×10^{4}	5,4	19.764	4.0382×10^{5}
2,2	8.4206	7.3303×10^{4}	0,5	5.2735	2.8750×10^{4}
3,2	11.345	1.3306×10^{5}	1,5	9.0049	8.3830×10^{4}
4,2	14.245	2.0978×10^{5}	2,5	12.138	1.5231×10^{5}
5,2	17.129	3.0332×10^{5}	3,5	15.155	2.3745×10^{5}
			4,5	18.117	3.3931×10^{5}
			5,5	21.042	4.5773×10^{5}

the root ε_0, and the solution corresponding to the complex conjugate roots ε_1 and $\bar{\varepsilon}_1$.† The real part of either ε_1 or $\bar{\varepsilon}_1$ gives the attenuation coefficient, and the corresponding imaginary part gives the angular frequency of the given nodal system.

Let s denote the number of nodal rings, and let n denote the number of nodal diameters. For the example

$$E = 3 \times 10^{10}, \qquad E_e = 2.6 \times 10^{10},$$
$$\nu = \nu_e = 0.33, \qquad \rho = 1.05$$

and for

$$\alpha = -100$$

we obtain for the (1,0) mode, i.e. $s = 1$ and $n = 0$

$$k = 2.76244836 + \mathrm{i}\,1.08708235 \times 10^{-16}$$
$$\varepsilon_0 = -86.66827563 + \mathrm{i}\,7.304885779 \times 10^{-11}$$
$$\varepsilon_1 = -6.665862186 + \mathrm{i}\,7.889078539 \times 10^{3}$$
$$\bar{\varepsilon}_1 = -6.665862186 - \mathrm{i}\,7.889078539 \times 10^{3}.$$

The preceding computed values clearly illustrate that k and ε_0 are real, and ε_1 and $\bar{\varepsilon}_1$ are complex conjugates. In Table 1, we list the values of k and the angular frequencies of the various modes computed for the particular example. The attenuation coefficients do not

† The solution corresponding to the root ε_0 is nonoscillatory. Henceforth, we shall only give the results corresponding to the oscillatory solution.

Table 2

E_e	ATTENUATION
1.6×10^{10}	-23.333
1.8×10^{10}	-20.000
2.0×10^{10}	-16.667
2.2×10^{10}	-13.333
2.4×10^{10}	-10.000
2.6×10^{10}	- 6.6667
2.7×10^{10}	- 5.0000
2.8×10^{10}	- 3.3333
3.0×10^{10}	0

Table 3

α	ATTENUATION
- 1	- 0.05000
-10	- 0.50000
-10^2	- 5.0000
-10^3	-50.000
-10^4	-4.9936×10^2
-10^5	-4.4151×10^3

change very much at all for the various modes. They vary from -6.6643 to -6.6667; the latter holds for the majority of the modes except for those of low frequencies. In addition, it may be readily verified that the values of k and the frequencies are in close agreement with those obtained via the corresponding elastic bending theory of thin plates, see e.g. Cowell and Hardy[6].

The values of the attenuation coefficient depend on the values of E_e and α. For the example

$$E = 3\times10^{10},\qquad \nu=\nu_e=0.33,$$
$$\rho=1.05,\qquad \alpha=-100$$

we list in Table 2 the values of the attenuation coefficient vs E_e for, say, the (5, 5) mode. In addition, for the example

$$E=3\times10^{10},\qquad E_e=2.7\times10^{10},$$
$$\nu=\nu_e=0.33,\qquad \rho=1.05$$

we list in Table 3 the values of the attenuation coefficient vs α for, say, the (4, 3) mode. It is of interest to note that these results are in keeping with physical notions. As E_e tends to E or as α tends to zero (equivalently the relaxation time τ, defined by eqn (5), tends to infinity) the material of the plate is said to exhibit less and less mechanical dissipation, and, therefore, the value of the attenuation coefficient tends to zero. It should also be noted that for the values of E_e and α of the preceding examples there does not seem to be any change in the value of k. The largest change in the frequency due to the values of E_e is a decrease in the fifth significant figure of the (0, 2) mode at $E_e = 1.6\times10^{10}$. At $\alpha=-10^5$, the values of the frequency decrease from a high of slightly over 5% for the (0, 2) mode to a low of within 1% for the (5, 5) mode.

Table 4

ATTENUATION COEFFICIENTS, ν_e=0.301						
s \ n	0	1	2	3	4	5
0			-4.3153	-4.5232	-4.6515	-4.7513
1	-6.6215	-6.0646	-5.8631	-5.7657	-5.7142	-5.6855
2	-6.0460	-5.9729	-5.9155	-5.8741	-5.8443	-5.8227
3	-5.9660	-5.9421	-5.9183	-5.8978	-5.8809	-5.8671
4	-5.9391	-5.9284	-5.9163	-5.9052	-5.8945	-5.8855
5	-5.9268	-5.9212	-5.9142	-5.9071	-5.9004	-5.8941

The preceding results correspond to examples which obey condition (31), namely the instantaneous and equilibrium Poisson's ratios are equal. Physically, we expect that the Poisson's ratios are *unequal* so that condition (31) is a special case which serves to illustrate some of the nature of the solutions.

Solutions of examples obeying condition (32) turn out to be most interesting. In general, the values of k for the various modes are complex, and the solution of the cubic eqn (25) yields a real root ε_0 and two complex roots ε_i, $i = 1, 2$, which are not complex conjugates. Further, only one of the three pairs, say (k, ε_1), satisfies the boundary conditions in the sense of eqn (29).† In this situation, there is *no* non-oscillatory solution. The real parts of k and the imaginary parts of ε_1 (i.e. the angular frequencies) are equal to the corresponding values given in Table 1. However, the real parts of ε_1 now depend on the nodal systems. In other words, the values of the attenuation coefficient are different for different modes.

Occasionally, when the value of ν_e is close to that of ν, we encounter situations for which k is nearly real and the two complex roots are nearly complex conjugates whose real parts agree to within four or five significant figures. In these situations we pick the root which satisfies the boundary conditions the best as the solution of a given nodal system. It should be noted that the pair (k, ε_0) still does not satisfy the boundary conditions. Here again, there is only one solution.

Consider the example

$$E = 3 \times 10^{10}, \qquad E_e = 2.7 \times 10^{10},$$
$$\nu = 0.33, \qquad \rho = 1.05,$$
$$\alpha = -1.00.$$

In Tables 4 and 5 we list, respectively, the real parts of ε_1, i.e. the attenuation coefficients, for $\nu_e = 0.301$ and 0.35. We should note that when $\nu_e = 0.301$ the value of k of the (4,3) mode is nearly real, and when $\nu_e = 0.35$ the values of k of the (0,2), (4,3) and (5,3) modes are nearly real. It should also be noted that we are not able to satisfy the boundary conditions very well for the (0,2) mode in the latter situation, the imaginary part of eqn (29) is zero to within 2%.

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† The conjugate pair $(\bar{k}, \bar{\varepsilon}_1)$ is also a solution of the problem.

Table 5

ATTENUATION COEFFICIENTS, ν_e=0.35						
s \ n	0	1	2	3	4	5
0			-4.3009	-5.3008	-5.2019	-5.1261
1	-3.8521	-4.1887	-4.3328	-4.4024	-4.4393	-4.4598
2	-4.2021	-4.2544	-4.2954	-4.3250	-4.3463	-4.3618
3	-4.2593	-4.2764	-4.2934	-4.3080	-4.3201	-4.3300
4	-4.2785	-4.2862	-4.2948	-4.3029	-4.3105	-4.3038
5	-4.2873	-4.2914	-4.2963	-4.3018	-4.3062	-4.3107

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